Serie 7 – Parte I - Problema 4:

Exprese la Transformada Z de $y(n)=\sum_{k=-\infty}^{n} x(k)$ en función de *X(z)*.

Puede verse que

$$y(n-1)=\sum_{k=-\infty}^{n-1} x(k) \Rightarrow y(n)-y(n-1)=\sum_{k=-\infty}^{n} x(k)-\sum_{k=-\infty}^{n-1} x(k)=x(n)$$

$$\Rightarrow Z\left[y(n)-y(n-1)\right]=X(z)$$

$$\Rightarrow Y(z)-z^{-1}Y(z)=X(z)$$

$$\Rightarrow \left(1-z^{-1}\right)Y(z)=X(z)$$

$$\Rightarrow Y(z)=\frac{1}{\left(1-z^{-1}\right)}X(z)$$

Serie 7 – Parte I - Problema 18:

Sea el sistema en TD con función transferencia Z

$$H(z)=\frac{z^{-1}+\frac{1}{2}z^{-2}}{1-\frac{3}{5}z^{-1}+\frac{2}{25}z^{-2}}$$

Determine:

a. La respuesta al impulso.

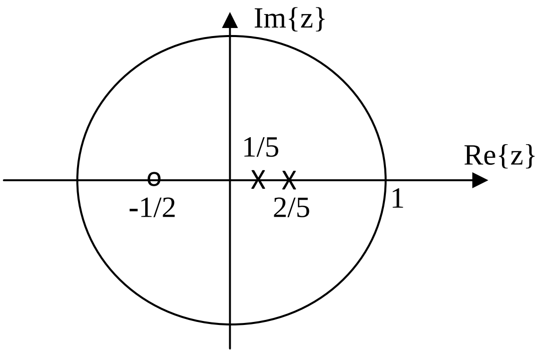
b. La respuesta al escalón con condiciones iniciales nulas.

c. La respuesta al escalón si $y(-1) = 1$ e $y(-2) = 2$.

a. Se verifica

$$H(z)=\frac{z^{-1}+\frac{1}{2}z^{-2}}{1-\frac{3}{5}z^{-1}+\frac{2}{25}z^{-2}}=\frac{z+\frac{1}{2}}{z^{2}-\frac{3}{5}z+\frac{2}{25}}$$

La FT tiene un cero en $c=-\frac{1}{2}$ y polos en $p_{1,2}=\frac{\frac{3}{5}\pm\sqrt{\frac{9}{25}-\frac{8}{25}}}{2}=\frac{\frac{3}{5}\pm\frac{1}{5}}{2}\Rightarrow\begin{cases}p_1=\frac{2}{5}\\ p_2=\frac{1}{5}\end{cases}$



La respuesta al impulso a partir de condiciones iniciales nulas resulta: $h(n) = Z^{-1}\{H(z)\}$

Para el cálculo debemos separar $H(z)$ en fracciones simples.

$$H(z) = \frac{z+\frac{1}{2}}{z^2 - \frac{3}{5}z + \frac{2}{25}} = \frac{z+\frac{1}{2}}{\left(z-\frac{1}{5}\right)\left(z-\frac{2}{5}\right)} = \frac{A}{\left(z-\frac{1}{5}\right)} + \frac{B}{\left(z-\frac{2}{5}\right)}$$

$$A = \lim_{z\to\frac{1}{5}} \frac{z+\frac{1}{2}}{z-\frac{2}{5}} = -\frac{\frac{7}{10}}{\frac{1}{5}} = -\frac{7}{2}$$

$$B = \lim_{z\to\frac{2}{5}} \frac{z+\frac{1}{2}}{z-\frac{1}{5}} = \frac{\frac{9}{10}}{\frac{1}{5}} = \frac{9}{2}$$

$$\Rightarrow H(z) = -\frac{7}{2}\frac{z^{-1}}{\left(1-\frac{1}{5}z^{-1}\right)} + \frac{9}{2}\frac{z^{-1}}{\left(1-\frac{2}{5}z^{-1}\right)}$$

$$\Rightarrow h(n) = -\frac{7}{2}\left(\frac{1}{5}\right)^{n-1}\mu(n-1) + \frac{9}{2}\left(\frac{2}{5}\right)^{n-1}\mu(n-1) \qquad (1)$$

b. Para una entrada escalón: $u(n) = \mu(n) \Rightarrow U(z) = \frac{1}{1-z^{-1}}$, luego

$$Y(z) = H(z)U(z) = \frac{z+\frac{1}{2}}{\left(z-\frac{1}{5}\right)\left(z-\frac{2}{5}\right)}\frac{z}{(z-1)}$$

$$\Rightarrow \frac{Y(z)}{z} = \frac{z+\frac{1}{2}}{\left(z-\frac{1}{5}\right)\left(z-\frac{2}{5}\right)(z-1)} = \frac{A}{\left(z-\frac{1}{5}\right)} + \frac{B}{\left(z-\frac{2}{5}\right)} + \frac{C}{(z-1)}$$

$$A = \lim_{z\to\frac{1}{5}} \frac{z+\frac{1}{2}}{\left(z-\frac{2}{5}\right)(z-1)} = \frac{\frac{7}{10}}{\frac{4}{25}} = \frac{35}{8}$$

$$B = \lim_{z\to\frac{2}{5}} \frac{z+\frac{1}{2}}{\left(z-\frac{1}{5}\right)(z-1)} = -\frac{\frac{9}{10}}{\frac{3}{25}} = -\frac{15}{2}$$

$$C = \lim_{z\to 1} \frac{z+\frac{1}{2}}{\left(z-\frac{1}{5}\right)\left(z-\frac{2}{5}\right)} = \frac{\frac{3}{2}}{\frac{12}{25}} = \frac{25}{8}$$

Luego

$$Y(z)=\frac{35}{8}\frac{z}{\left(z-\frac{1}{5}\right)}-\frac{15}{2}\frac{z}{\left(z-\frac{2}{5}\right)}+\frac{25}{8}\frac{z}{(z-1)}$$

$$\Rightarrow y(n)=\frac{35}{8}\left(\frac{1}{5}\right)^{n}\mu(n)-\frac{15}{2}\left(\frac{2}{5}\right)^{n}\mu(n)+\frac{25}{8}\mu(n) \qquad (2)$$

c. Se tiene que

$$H(z)=\frac{z^{-1}+\frac{1}{2}z^{-2}}{1-\frac{3}{5}z^{-1}+\frac{2}{25}z^{-2}}=\frac{Y(z)}{U(z)}\Rightarrow Y(z)\left(1-\frac{3}{5}z^{-1}+\frac{2}{25}z^{-2}\right)=U(z)\left(z^{-1}+\frac{1}{2}z^{-2}\right)$$

$$\Rightarrow y(n)-\frac{3}{5}y(n-1)+\frac{2}{25}y(n-2)=u(n-1)+\frac{1}{2}u(n-2)$$

Ecuación en diferencias (3)

Para poder incorporar las condiciones iniciales debemos transformar Z la ecuación en diferencias considerando la propiedad de Retardo Temporal de la transformada Z unilateral:

$$Z\left[x(n-k)\right]=z^{-k}\left[\;X(z)+\sum_{n=1}^{k}x(-n)z^{n}\;\right]\quad,\ k>0$$

Resulta entonces:

$$Y(z)-\frac{3}{5}z^{-1}\left(Y(z)+y(-1)z\right)+\frac{2}{25}z^{-2}\left(Y(z)+y(-1)z+y(-2)z^{2}\right)=U(z)\left(z^{-1}+\frac{1}{2}z^{-2}\right)$$

$$Y(z)\left(1-\frac{3}{5}z^{-1}+\frac{2}{25}z^{-2}\right)-\frac{3}{5}y(-1)+\frac{2}{25}y(-1)z^{-1}+\frac{2}{25}y(-2)=U(z)\left(z^{-1}+\frac{1}{2}z^{-2}\right)$$

$$Y(z)\left(1-\frac{3}{5}z^{-1}+\frac{2}{25}z^{-2}\right)=\left(\frac{3}{5}-\frac{2}{25}z^{-1}\right)y(-1)-\frac{2}{25}y(-2)+U(z)\left(z^{-1}+\frac{1}{2}z^{-2}\right)$$

$$Y(z)=\underbrace{\frac{\left(\frac{3}{5}-\frac{2}{25}z^{-1}\right)}{\left(1-\frac{3}{5}z^{-1}+\frac{2}{25}z^{-2}\right)}y(-1)-\frac{\frac{2}{25}}{\left(1-\frac{3}{5}z^{-1}+\frac{2}{25}z^{-2}\right)}y(-2)}_{\text{respuesta libre}}+\underbrace{\underbrace{\frac{\left(z^{-1}+\frac{1}{2}z^{-2}\right)}{\left(1-\frac{3}{5}z^{-1}+\frac{2}{25}z^{-2}\right)}}_{H(z)}U(z)}_{\text{respuesta forzada}} \qquad (4)$$

respuesta libre **respuesta forzada**

Para una entrada escalón, la respuesta forzada fue calculada en b. (ecuación (2)).
Calculemos la respuesta libre (a condiciones iniciales).

$$Y_{\text{libre}}(z)=\frac{\left(\frac{3}{5}-\frac{2}{25}z^{-1}\right)}{\left(1-\frac{3}{5}z^{-1}+\frac{2}{25}z^{-2}\right)}\underbrace{y(-1)}_{=1}-\frac{\frac{2}{25}}{\left(1-\frac{3}{5}z^{-1}+\frac{2}{25}z^{-2}\right)}\underbrace{y(-2)}_{=2}$$

$$\frac{Y_{\text{libre}}(z)}{z}=\frac{\left(\frac{3}{5}z-\frac{2}{25}\right)}{\left(z^2-\frac{3}{5}z+\frac{2}{25}\right)}-\frac{\frac{4}{25}z}{\left(z^2-\frac{3}{5}z+\frac{2}{25}\right)}$$

$$\frac{Y_{\text{libre}}(z)}{z}=\frac{\left(\frac{11}{25}z-\frac{2}{25}\right)}{\left(z^2-\frac{3}{5}z+\frac{2}{25}\right)}=\frac{\left(\frac{11}{25}z-\frac{2}{25}\right)}{\left(z-\frac{1}{5}\right)\left(z-\frac{2}{5}\right)}=\frac{A}{\left(z-\frac{1}{5}\right)}+\frac{B}{\left(z-\frac{2}{5}\right)}$$

Fracciones simples

$$A=\lim_{z\to\frac{1}{5}}\frac{\left(\frac{11}{25}z-\frac{2}{25}\right)}{\left(z-\frac{2}{5}\right)}=-\frac{1}{25}$$

$$B=\lim_{z\to\frac{2}{5}}\frac{\left(\frac{11}{25}z-\frac{2}{25}\right)}{\left(z-\frac{1}{5}\right)}=\frac{1}{25}$$

$$\Rightarrow Y_{\text{libre}}(z)=-\frac{1}{25}\frac{1}{\left(1-\frac{1}{5}z^{-1}\right)}+\frac{1}{25}\frac{1}{\left(1-\frac{2}{5}z^{-1}\right)}$$

$$\Rightarrow y_{\text{libre}}(n)=-\frac{1}{25}\left(\frac{1}{5}\right)^n\mu(n)+\frac{1}{25}\left(\frac{2}{5}\right)^n\mu(n) \qquad (5)$$

Respuesta Total = (2) + (5)